Table 2 Out-of-plane stresses and electric displacement

	Applied potential		Applied load	
Height	$\sigma_z \times 10^3$	$\sigma_{xz} \times 10^3$	$\sigma_z \times 10^1$	$D_z \times 10^{13}$
0.500	0.0000	0.0000	10.000	160.58
0.475	−0.8333	41.457	9.9657	149.35
0.450	−2.8471	64.626	9.8682	117.23
0.425	−5.3241	69.556	9.7154	66.568
0.400	−7.5482	56.259	9.5151	−0.3382
0.300	−12.957	19.082	8.5199	−0.1276
0.200	−15.245	−4.5693	7.3747	0.0813
0.100	−15.510	−18.203	6.1686	0.2913
0.000	−14.612	−23.866	4.9831	0.5052
−0.100	−12.524	−25.282	3.8045	0.7259
−0.200	−9.2558	−25.633	2.6137	0.9563
−0.300	−5.5018	−24.994	1.4821	1.1995
−0.400	−1.8733	−23.379	0.4868	1.4587
−0.425	−1.1074	−18.888	0.2845	−58.352
−0.450	−0.5162	−13.501	0.1312	−103.66
−0.475	−0.1351	−7.2092	0.0340	−132.40
−0.500	0.0000	0.0000	0.0000	−142.46

Table 3 In-plane stresses

	Applied potential		Applied load	
Height	$\sigma_x \times 10^2$	$\sigma_{xy} \times 10^2$	σ_x	σ_{xy}
0.500	111.81	−146.03	6.5643	−2.4766
0.475	63.736	−100.77	5.8201	−2.1824
0.450	15.833	−55.693	5.0855	−1.8942
0.425	−32.001	−10.698	4.3595	−1.6114
0.400	−79.865	34.295	3.6408	−1.3332
0.400	−51.681	6.3365	2.8855	−0.2463
0.300	−33.135	4.6631	1.4499	−0.1534
0.200	−19.840	3.3247	0.2879	−0.0817
0.100	−9.7737	2.2096	−0.7817	−0.0212
0.000	−1.3905	1.2286	−1.9266	0.0369
0.000	−1.3089	1.2287	0.0991	0.0369
−0.100	−0.5782	0.5227	−0.0149	0.0965
−0.200	0.1348	−0.0572	−0.1280	0.1529
−0.300	0.8463	−0.5840	−0.2426	0.2139
−0.400	1.5723	−1.1220	−0.3616	0.2882
−0.400	14.529	−6.0731	−4.2348	1.5603
−0.425	178.01	−7.3455	−4.8806	1.8105
−0.450	210.98	−8.6346	−5.5337	2.0651
−0.475	244.28	−9.9437	−6.1951	2.3246
−0.500	277.95	−11.276	−6.8658	2.5899

and $\epsilon_{11}/\epsilon_0 = \epsilon_{22}/\epsilon_0 = 1475$, $\epsilon_{33}/\epsilon_0 = 1300$. The piezoelectric layer thicknesses are taken as 0.1 m.

Both applied double sinusoidal loading and surface potentials are considered, with $m = n = 1$ and $q_0 = \phi_0 = 1$. The units of these constants and the resulting elastic and electric field quantities are consistent with those given for the material properties. The aspect ratio in both cases is $L/h = 4$. For the applied load, the top and bottom laminate surfaces are fixed at zero potential. For the applied potential, these surfaces are stress free.

Representative through-thickness distributions are shown in Tables 1–3. The in-plane stress distributions reflect the discontinuous nature at each layer interface. The midplane transverse displacements are -14.711×10^{-12}m for the applied potential and 30.027×10^{-11}m for the applied load.

Closure

The present formulation can be applied to bonded laminates of dissimilar piezoelectric materials, which also require continuity of the eight elastic and electric field variables used here. Embedded piezoelectric layers and specified internal quantities are easily incorporated into the analysis.

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Dynamic Response of Cross-Ply Shallow Shells with Levy-Type Boundary Conditions

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I. Introduction

THE analysis of laminated composite shells has been the subject of significant research interest in recent years. The classical lamination shell theories based on the Love-Kirchhoff assumptions are adequate to predict the gross behavior of thin laminates. When the structures are rather thick or when they exhibit high anisotropy ratios, the transverse shear deformation effect has to be incorporated. In such cases more refined theories are needed. The third-order theory used in the present study is proposed by Reddy and Liu.[1] Closed-form solutions for the dynamic response of laminated shells have been developed mainly for the case of simply supported boundary conditions. Ritz, Galerkin, and other approximate methods are used for other boundary conditions. The need for analytical solutions for the dynamic response of composite laminates for a variety of boundary conditions is worthy to be mentioned. In the present work, a generalized modal approach in conjunction with the Levy method is presented to solve for the transient response of cross-ply laminated shallow shells with various boundary conditions and for arbitrary loadings. To demonstrate the method, I present numerical results of theories for center deflections of spherical shells subjected to sinusoidal loading in the spatial domain and sine pulse loading in the time domain.

Analytical Solution

A generalized modal approach is used to solve the equations of motion of laminated composite shallow shells with Levy-type boundary conditions. The edges $x_2 = 0, b$ are assumed simply supported, whereas the remaining ones ($x_1 = \pm a/2$) may have arbitrary combinations of free, clamped, and simply supported edge conditions. In this approach, we express the generalized displacements as products of undetermined functions and known trigonometric functions so as to satisfy the simply supported boundary conditions at $x_2 = 0, b$. The equations of motion can be reduced in the following state space equation by defining a state vector $\{y(x_1, t)\}$

$$\{y'\} = [M]\{\ddot{y}\} + [K]\{y\} + \{r\} \tag{1}$$

where a prime and dots on a quantity denote the derivative with respect to x_1 and time t, respectively; $\{r\}$ is the load vector. In the case of a free vibration problem, the vector $\{y\}$ will be separated

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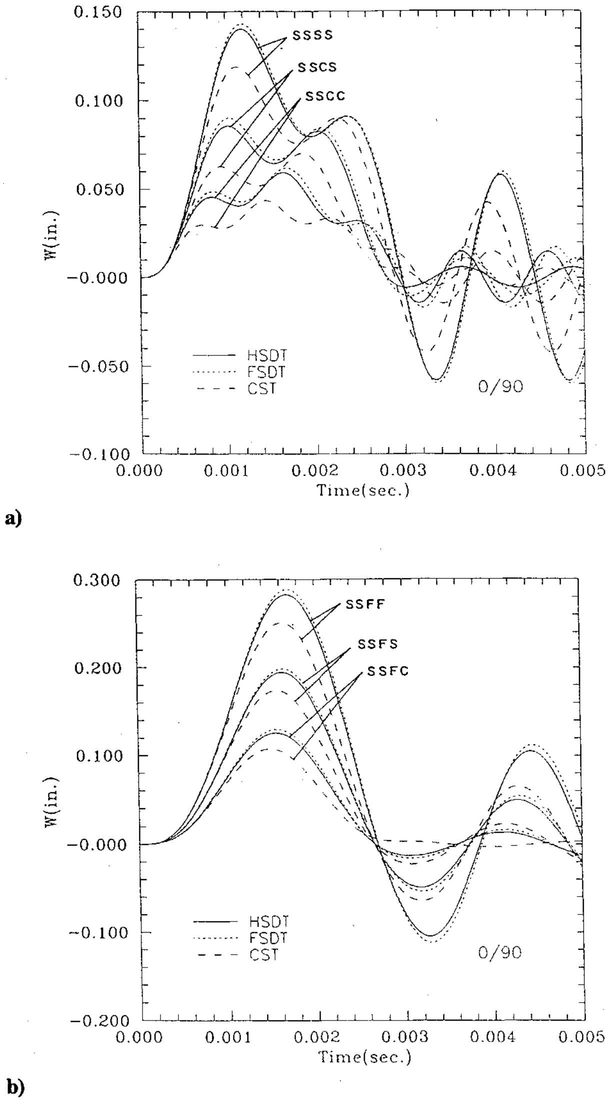

Fig. 1 Transient response of two-layered (0/90) spherical shells for various boundary conditions: a) SSSS, SSCS, and SSCC and b) SSFF, SSFS, and SSFC.

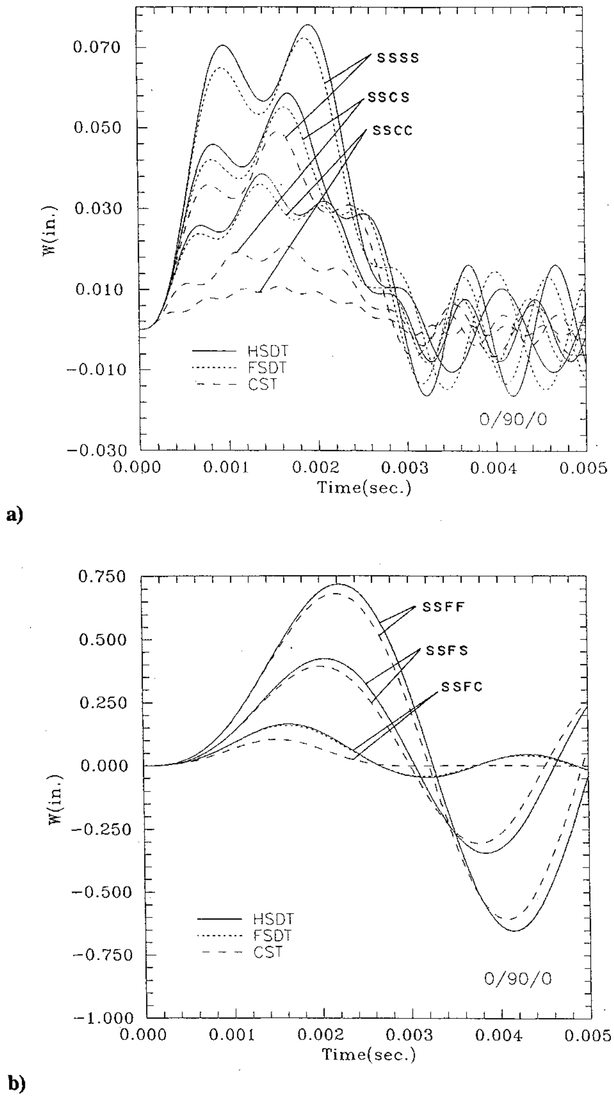

Fig. 2 Transient response of three-layered (0/90/0) spherical shells for various boundary conditions: a) SSSS, SSCS, and SSCC and b) SSFF, SSFS, and SSFC.

into time and spatial coordinates as

$$\{y\} = \{Y_m(x_1)\}T_m(t) \tag{2}$$

To obtain the frequencies and the corresponding eigenfunctions, the generalized coordinates $T_m(t)$ must satisfy

$$\ddot{T}_m + \omega_m^2 T_m = 0 \tag{3}$$

and the eigenfunctions $\{Y_m\}$ will fulfill the following equation:

$$\{Y'\} = [A]\{Y\} \tag{4}$$

where

$$[A] = [K] - \omega_m^2[M] \tag{5}$$

and where ω_m is the natural frequency corresponding to the mth mode. There are infinite frequencies for each value of m and the dynamic response is governed mainly by the fundamental frequency of each mode.

The solution to Eq. (4) is given by

$$\{Y(x_1)\} = [D]\begin{bmatrix} e^{\lambda_1 x_1} & & \underline{0} \\ & \ddots & \\ & & \ddots \\ \underline{0} & & e^{\lambda_n x_1} \end{bmatrix}\{l\} \tag{6}$$

where λ_i are the distinct eigenvalues of the matrix $[A]$, whereas $[D]$ denotes the matrix of eigenvectors of $[A]$. Substitution of Eq. (6) into the boundary conditions associated with the edges $x_1 = \pm a/2$ results in a set of homogeneous algebraic equations of the form

$$[B][D]^{-1}\{k\} = \{0\} \tag{7}$$

For a nontrivial solution of Eq. (7), the determinant should be zero

$$|B|/|D| = 0 \tag{8}$$

Equations (8) and (6) give the eigenfrequencies and the associated eigenfunctions, respectively. Equation (1) is not a self-adjoint equation, and the eigenfunctions do not form an orthogonal set; therefore we must obtain the eigenfunction of the adjoint of Eq. (4)

to decouple Eq. (1). The adjoint of Eq. (4) is

$$\{Z'\} = -[A]^T\{Z\} \tag{9}$$

with the following associated boundary conditions:

$$\{Z\}^T\{Y\}\Big|_{-a/2}^{a/2} = 0 \tag{10}$$

A formal solution of Eq. (9) is given by

$$\{Z\} = [C]\begin{bmatrix} e^{-\lambda_1 x_1} & & \underline{0} \\ & \ddots & \\ & & \ddots \\ \underline{0} & & e^{-\lambda_n x_1} \end{bmatrix}\{n\} \tag{11}$$

where $[C]$ denotes the matrix of eigenvectors of $-[A]^T$.

Substitution of Eq. (11) into the corresponding boundary conditions of the adjoint problem at the edges $x_1 = \pm a/2$ results in a homogeneous algebraic equation of the form

$$[E]\{n\} = 0 \tag{12}$$

We have to solve for the eigenvector $\{n\}$ corresponding to each frequency ω.

Making use of the following biorthogonality conditions of the natural modes with respect to the eigenfunctions $\{Y_m\}$ and $\{Z_n\}$,

$$-\int_{-a/2}^{a/2}\{Z_n\}^T[M]\{Y_m\}\,\mathrm{d}x_1 = M_m\delta_{mn} \tag{13}$$

$$\int_{-a/2}^{a/2}\{Z_n\}^T(\{Y_m'\} - [K]\{Y_m\})\,\mathrm{d}x_1 = \omega_m^2 M_m\delta_{mn} \tag{14}$$

and substituting Eq. (2) into Eq. (1), performing left multiplication by the adjoint eigenfunction $\{Z_n\}^T$, and integrating over the domain, we obtain

$$\ddot{T}_m(t) + \omega_m^2 T_m(t) = \frac{1}{M_m}\int_{-a/2}^{a/2}\{Z_m\}^T\{r_m\}\,\mathrm{d}x_1 \ . \tag{15}$$

For zero initial conditions, the state vector $\{y\}$ will be expressed as

$$\{y_m(x_1,t)\} = \frac{1}{M_m}\{Y_m(x_1)\}\int_0^t h_m(t-\tau) \times \int_{-a/2}^{a/2}\{Z_m\}^T\{r_m(\xi,\tau)\}\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{16}$$

where $h_m(t-\tau)$ is the impulse response function.

Numerical Results and Discussion

The numerical applications are carried out for cross-ply spherical shells whose geometrical and material properties are the same for all layers. The transverse deflection presented in the figures is evaluated at $(x_1, x_2, \zeta) = (0, b/2, \zeta)$. Zero initial conditions are assumed. The variations of center deflection with time for antisymmetric cross-ply (0/90) and symmetric cross-ply (0/90/0) laminated spherical caps are shown in Figs. 1 and 2, respectively, for various boundary conditions. It is interesting to note that for SSSS, SSCS, and SSCC boundary conditions the amplitudes are smaller for symmetric cross-ply than for antisymmetric cross-ply laminates, whereas for SSFF, SSFS, and SSFC the amplitudes are higher. Moreover, the first-order (FSDT) and the third-order (HSDT) theories predict almost the same response, whereas the classical theory (CST) differs both in amplitude and phase.

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Vibration of Clamped Right Triangular Thin Plates: Accurate Simplified Solutions

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Nomenclature

a = plate dimension in x direction
b = plate dimension in y direction
D = flexural rigidity of plate, $Eh^3/12(1-\nu^2)$
E = Young's modulus of plate material
h = plate thickness
k = number of terms used in solution
k^* = upper limit for first summations of solutions
W = plate lateral displacement divided by side length a
η = distance along plate y axis divided by side length b, y/b
λ^2 = eigenvalue, $\omega a^2\sqrt{\rho/D}$
ν = Poisson's ratio of plate material
ξ = distance along plate x axis divided by side length a, x/a
ρ = mass of plate per unit area
ϕ = plate aspect ratio, b/a
ω = circular frequency of plate vibration

I. Introduction

USE of the superposition techniques in the free-vibration analyses of thin plates, as they were first introduced by Gorman,[1] has provided simple and effective solutions to a vast number of rectangular plate problems. The method has also been extended to nonrectangular plates such as triangular and trapezoidal plates. However, serious difficulties were encountered in some of these analyses. These difficulties were discussed and obviated in Ref. 2. This reference, however, dealt only with simple support conditions, leading to a simple, highly accurate, and very economical solution to the free-vibration problem of simply supported right angle triangular plates.

The purpose of this Note is to show that the modified superposition method of Ref. 2 is also applicable to clamped-edge conditions. This is accomplished through the application of this method to the title problem.

II. Mathematical Procedure

The solution is similar in every aspect to that of the simply supported right angle triangular plate of Ref. 2. It is based on the principles of superposition. In this method, a number of appropriate building blocks, for which Lévy-type solutions are readily available or easily obtainable, are superimposed. The contributions of these building blocks to the various boundary conditions of the original plate are formulated. Finally, the coefficients appearing in these formulations are adjusted to satisfy the prescribed edge conditions of the plate under study. Depending on its aspect ratio, the right angle

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